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The harmonic oscillations of a viscoelastic rod of triangular cross-section☆

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ABSTRACT

Two exact solutions of the plane strain problem of the harmonic oscillations of a viscoelastic rod, the cross-section of which is a right triangle, are proposed. Either the normal displacement and the shear stress or the shear displacement and the normal stress of the side surface of the rod are given. Six dimensionless parameters which affect the dynamic deformation process are derived. Two parameters characterize the contribution of the viscous properties with respect to the elastic properties, two others define the logarithmic decrement of the longitudinal and shear harmonic waves, and two other parameters affect the wavelength of the corresponding wave and the velocity of motion of the wave front of these waves. The velocities of both types of waves and their wavelengths turn out to be greater than the velocities and wavelengths of the corresponding elastic waves. It is shown that, for certain values of the viscosity and the oscillation frequency, pseudo-resonance frequencies are possible which are higher than the resonance frequencies for an elastic medium.

Several exact solutions of one-dimensional classical problems of the theory of viscoelasticity are known[1–3] in a dynamic formulation. Exact solutions of two-dimensional dynamic problems are proposed below using special variables, which were used for the first time in Ref. 4 to solve the Stefan phase-transition problem. The possibility of using this approach for elastic materials was pointed out previously (see Ref. 5 etc.).

1. Formulation of the problem

For a viscoelastic body, we will use the Kelvin rheological model,[6] for which the components of the stress tensor σ_{ij} are expressed in terms of the components of the strain tensor d_{ij} and the strain rate tensor ε_{ij} as follows:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \zeta\varepsilon_{kk}\delta_{ij} + 2\eta\varepsilon_{ij} \tag{1.1}$$

From the equations of motion of a continuous medium in Cartesian coordinates (x,y), we obtain the following two differential equations in the displacements u and v for plane deformation

$$\lambda_0 u_{xx} + (\lambda+\mu)\upsilon_{xy} + \mu u_{yy} + \zeta_0 u_{txx} + (\zeta+\eta)\upsilon_{txy} + \eta u_{tyy} = \rho u_{tt} \quad (u \leftrightarrow \upsilon,\ x \leftrightarrow y)$$
$$\lambda_0 = \lambda + 2\mu, \quad \zeta_0 = \zeta + 2\eta \tag{1.2}$$

Here and henceforth notation of the form $(u \leftrightarrow v, x \leftrightarrow y)$ denotes that one further similar unwritten relation can be obtained by simultaneous inversion of the quantities in parenthesis.

We will consider the problem without initial conditions on the harmonic oscillations of a viscoelatic rod, the cross-section of which Ω is a right triangle of height $2h$, which occurs as a result of periodic actions on its side boundary Γ. Two versions of the boundary conditions are specified.

Version 1. On the rod surface Γ the normal displacement u_n and the shear stress τ_n are given:

$$u_n|_\Gamma = u_{10}\cos\omega t + u_{20}\sin\omega t, \quad \tau_n|_\Gamma = \tau_{10}\cos\omega t + \tau_{20}\sin\omega t \tag{1.3}$$

Version 2. The tangential displacement u_τ and the normal stress σ_n are specified on Γ:

$$u_\tau|_\Gamma = \upsilon_{10}\cos\omega t + \upsilon_{20}\sin\omega t,$$
$$\sigma_n|_\Gamma = \sigma_{10}\cos\omega t + \sigma_{20}\sin\omega t \tag{1.4}$$

Here u_{j0}, τ_{j0}, v_{j0} and σ_{j0} $(j = 1,2)$ are given constants.

We will seek a solution of the problem in the form

$$u = U_1(x,y)\cos\omega t + U_2(x,y)\sin\omega t,$$
$$\upsilon = V_1(x,y)\cos\omega t + V_2(x,y)\sin\omega t \tag{1.5}$$

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We then have the following system of four equations from the two equations (1.2)

$$\lambda_0 U_{jxx} + (\lambda+\mu)V_{jxy} + \mu U_{jyy} + \omega\zeta_0 U_{(3-j)xx} + \omega(\zeta+\eta)V_{(3-j)xy} + \omega\eta U_{(3-j)yy} + \rho\omega^2 U_j = 0 \quad (U \leftrightarrow V, x \leftrightarrow y), \quad j = 1, 2 \tag{1.6}$$

2. The solution for a plane viscoelastic strip

To solve the problem of the oscillations of a rod of triangular cross-section it is first necessary to consider the simpler similar problem for a plane viscoelastic strip. We will assume that U_j and V_j (j = 1, 2) depend only on one coordinate x, and we will therefore introduce the following notation

$$U_j = P_j(x), \quad V_j = Q_j(x), \quad j = 1, 2 \tag{2.1}$$

After substituting expressions (2.1) into (1.6) we arrive at a system of ordinary differential equations

$$\lambda_0 P_1'' + \omega\zeta_0 P_2'' + \rho\omega^2 P_1 = 0, \quad \lambda_0 P_2'' - \omega\zeta_0 P_1'' + \rho\omega^2 P_2 = 0 \tag{2.2}$$

$$\mu Q_1'' + \omega\eta Q_2'' + \rho\omega^2 Q_1 = 0, \quad \mu Q_2'' - \omega\eta Q_1'' + \rho\omega^2 Q_2 = 0 \tag{2.3}$$

From the characteristic equation of this system we can obtain all eight roots α_k and β_k (k = 1, . . ., 4) in explicit form

$$\alpha_{1,2} = \pm(m_0 - in_0), \quad \alpha_{3,4} = \pm(m_0 + in_0)$$
$$m_0 = \sqrt{\frac{\rho}{2}\frac{\omega}{\Lambda_0}}\sqrt{\Lambda_0 - \lambda_0}, \quad n_0 = \sqrt{\frac{\rho}{2}\frac{\omega}{\Lambda_0}}\sqrt{\Lambda_0 + \lambda_0},$$
$$\Lambda_0 = \sqrt{\lambda_0^2 + \omega^2\zeta_0^2} \tag{2.4}$$

$$\beta_{1,2} = \pm(p_0 - iq_0), \quad \beta_{3,4} = \pm(p_0 + iq_0)$$
$$p_0 = \sqrt{\frac{\rho}{2}\frac{\omega}{G_0}}\sqrt{G_0 - \mu}, \quad q_0 = \sqrt{\frac{\rho}{2}\frac{\omega}{G_0}}\sqrt{G_0 + \mu},$$
$$G_0 = \sqrt{\mu^2 + \omega^2\eta^2} \tag{2.5}$$

Bearing boundary conditions (1.3) and (1.4) in mind, the general solution of Eqs (2.2), (2.3) can be represented by the expressions

$$P_j(x) = [C_j \cos n_0(x-h) - (-1)^j C_{3-j} \sin n_0(x-h)] \exp[m_0(x-h)] + [C_{5-j} \sin n_0(x-h) - (-1)^j C_{j+2} \cos n_0(x-h)] \exp[m_0(h-x)]$$
$$(P_j(x) \leftrightarrow Q_j(x), C_j \leftrightarrow D_j, m_0 \leftrightarrow p_0, n_0 \leftrightarrow q_0), \; j = 1, 2 \tag{2.6}$$

The constants $C_1, \ldots, C_4$ and $D_1, \ldots, D_4$ can be obtained from the conditions on the strip boundaries, which leads to the solution of a closed system of inhomogeneous algebraic equations and presents no difficulties.

3. The solution for a viscoelastic rod of triangular cross-section for the first version of the boundary conditions

We will introduce the new geometric variable ξ and three further variables ξ_k by the formulae

$$\xi = (\mathbf{r} - \mathbf{r}_0)\mathbf{n}, \quad \xi_k = (\mathbf{r} - \mathbf{r}_k)\mathbf{n}_k, \quad k = 1, 2, 3 \tag{3.1}$$

where $\mathbf{n} = (\cos\theta, \sin\theta)$ is a certain unit vector, at an angle θ to the x axis, where θ is independent of the x and y coordinates and the time t, $\mathbf{n}_k$ are the inward unit normals to the sides of the triangle Ω, $\mathbf{r}_0$ is the radius vector of some pole, $\mathbf{r}_k$ is the radius vector of the vertex of the triangle, and $\mathbf{r}$ is the radius vector of an arbitrary point of the region Ω. With this definition of the variables ξ_k the equations of the sides of the rectangle will be given by the equalities $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$. For the points $(x, y) \in \Omega$ we have the strict inequalities $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$. The variables ξ and ξ_k and the normals $\mathbf{n}_k$ in the (x,y) plane possess the following properties, which we will need to use later:

$$\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 = 0, \quad \mathbf{n}_1\mathbf{n}_2 = \mathbf{n}_1\mathbf{n}_3 = \mathbf{n}_2\mathbf{n}_3 = -1/2 \tag{3.2}$$

$$\mathbf{n}_1 \times \mathbf{n}_2|_z = \mathbf{n}_2 \times \mathbf{n}_3|_z = \mathbf{n}_3 \times \mathbf{n}_1|_z = \sqrt{3}/2,$$
$$\xi_1 + \xi_2 + \xi_3 = 2h \tag{3.3}$$

$$F_x = F'(\xi)n_x, \quad F_y = F'(\xi)n_y, \quad F_{xx} = F''(\xi)n_x^2$$
$$F_{xy} = F''(\xi)n_x n_y, \quad F_{yy} = F''(\xi)n_y^2 \text{ When } F = F(\xi) \in C^2(\Omega) \tag{3.4}$$

Here $\mathbf{n}_j \times \mathbf{n}_{k|z}$ is the unique non-zero projection of the vector product onto the z axis.

Using the functions $P_j(\xi)$ and $Q_j(\xi)$, obtained from formulae (2.6), we can construct the following particular solution of system (1.6)

$$U_j(x, y) = P_j(\xi)n_x - Q_j(\xi)n_y,$$
$$V_j(x, y) = P_j(\xi)n_y + Q_j(\xi)n_x; \quad j = 1, 2 \tag{3.5}$$

Using properties (3.2)-(3.4) it can be proved that U_j and V_j satisfy system (1.6), since $P_j(\xi)$ and $Q_j(\xi)$ satisfy Eqs. (2.2) and (2.3). If we now replace the variable ξ on the right-hand sides of expressions (3.5) by any of the variables ξ_k, defined by the last equalities of (3.1), the expressions for U_j and V_j then obtained will also satisfy system (1.6). Using this property, the solution of problem (1.2), (1.3) can be represented by the following sums

$$U_j(x, y) = \sum_{k=1}^{3} [P_j(\xi_k)n_{kx} - Q_j(\xi_k)n_{ky}],$$
$$V_j(x, y) = \sum_{k=1}^{3} [P_j(\xi_k)n_{ky} + Q_j(\xi_k)n_{kx}];$$
$$j = 1, 2 \tag{3.6}$$

We will assume that when each ξ_k is substituted instead of x, the constants C_j and D_j in expressions (2.6) for $P_j(\xi)$ and $Q_j(\xi)$ will remain unchanged, and hence the functions in Eq. (3.6) contain 8 arbitrary constants in all, which must be obtained from the boundary conditions. To do this it is necessary to convert boundary conditions (1.3). We will write the normal component of the displacements on the sides of the triangle in the form

$$u_n|_\Gamma = (un_x + \upsilon n_y)|_\Gamma \quad \text{or} \quad (U_j n_x + V_j n_y)|_\Gamma = u_{j0}, \quad j = 1, 2 \tag{3.7}$$

In these problems we will assume that all the analytical relations with respect to the sides of the right triangle are equally justified, and hence all the boundary conditions are sufficiently satisfied on any one side, for example, on the side $\xi_3 = 0$. Then, the boundary conditions will be automatically satisfied on the other two sides of the triangle when $\xi_1 = 0$ or $\xi_2 = 0$. For points (x,y) on the side of the triangle $\xi_3 = 0$ we will have the following relation between the variables ξ_1 and ξ_2

$$\xi_1 + \xi_2 = 2h \text{ when } \xi_3 = 0 \tag{3.8}$$

After substituting expressions (3.6) into (3.7) we obtain the following two equations

$$[P_j(\xi_1)\mathbf{n}_1\mathbf{n}_3 + P_j(\xi_2)\mathbf{n}_2\mathbf{n}_3 + P_j(\xi_3) + Q_j(\xi_1)\mathbf{n}_1 \times \mathbf{n}_3|_z + \\ + Q_j(\xi_2)\mathbf{n}_2 \times \mathbf{n}_3|_z]|_{\xi_3 = 0} = u_{j0}, \quad \xi_2 = 2h - \xi_1, \quad j = 1, 2 \tag{3.9}$$

which, using properties (3.2) and (3.3), can be reduced to the form

$$\frac{\sqrt{3}}{2}[Q_j(2h-\xi_1) - Q_j(\xi_1)] - \frac{1}{2}[P_j(2h-\xi_1) + P_j(\xi_1)] \\ + P_j(0) = u_{j0}, \quad j = 1, 2 \tag{3.10}$$

The boundary conditions in the form (3.7) must be satisfied for any values of $\xi_1 \in [0, 2h]$, which is possible if the expressions in square brackets are equal to zero:

$$P_j(2h-\xi_1) + P_j(\xi_1) = 0, \\ Q_j(2h-\xi_1) - Q_j(\xi_1) = 0, \quad j = 1, 2 \tag{3.11}$$

Hence, using expressions (2.6) for P_j and Q_j, we obtain the following relation between the coefficients:

$$C_3 = -C_1, \quad C_4 = C_2, \quad D_3 = D_1, \quad D_4 = -D_2 \tag{3.12}$$

Here, Eq. (3.10) take the form of two linear algebraic equations

$$P_j(0) = u_{j0}, \quad j = 1, 2 \tag{3.13}$$

Hence, the first boundary condition of (1.3), reduced to the form (3.7), will be satisfied if the coefficients C_j and D_j are connected by relations (3.12) and (3.13).

Boundary condition (1.3) for the shear stress, according to equality (1.1), can be written as

$$\tau_n|_\Gamma = 2\mu\gamma_n|_\Gamma + 2\eta\frac{\partial\gamma_n}{\partial t}\bigg|_\Gamma \tag{3.14}$$

where

$$2\gamma_n|_\Gamma = \left(\frac{\partial u_\tau}{\partial n} + \frac{\partial u_n}{\partial \tau}\right)\bigg|_\Gamma, \quad 2\frac{\partial}{\partial t}\gamma_n\bigg|_\Gamma = \left(\frac{\partial}{\partial n}\frac{\partial u_\tau}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial u_n}{\partial t}\right)\bigg|_\Gamma \tag{3.15}$$

If the normal direction on Γ is defined by the unit vector $\mathbf{n} = (n_x, n_y)$, the tangential component of the displacement vector on Γ can be represented by the equation

$$u_\tau|_\Gamma = (un_y - \upsilon n_x)|_\Gamma$$

Since, in boundary conditions (1.3) the normal component u_n on Γ is specified by a boundary that is constant over the points, the expressions for the shear γ_n and the shear rate $\delta\gamma_n/\delta t$, defined by formulae (3.15), can be simplified, and boundary condition (1.3) for τ_n reduces to the form

$$\mu\left(\frac{\partial u}{\partial n}n_y - \frac{\partial \upsilon}{\partial n}n_x\right)\bigg|_\Gamma + \eta\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial n}n_y - \frac{\partial \upsilon}{\partial n}n_x\right)\bigg|_\Gamma = \tau_{10}\cos\omega t + \tau_{20}\sin\omega t \tag{3.16}$$

Using representation (1.5) and introducing the notation

$$T_j = \frac{\partial}{\partial n_3}(U_j n_{3y} - V_j n_{3x})|_{\xi_3 = 0}, \quad j = 1, 2 \tag{3.17}$$

from equality (3.16) in T_j we will have the following system of two equations

$$\mu T_1 + \eta\omega T_2 = \tau_{10}, \quad \mu T_2 - \eta\omega T_1 = \tau_{20} \tag{3.18}$$

from which we obtain

$$T_j = \frac{\mu\tau_{j0} + (-1)^j\eta\omega\tau_{(3-j)0}}{\mu^2 + \eta^2\omega^2}, \quad j = 1, 2 \tag{3.19}$$

If we substitute expressions (3.6) for U_j and V_j into the right-hand side of the boundary conditions in the form (3.17), these conditions will take the form

$$\frac{\partial}{\partial n_3}[P_j(\xi_1)\mathbf{n}_1 \times \mathbf{n}_3|_z + P_j(\xi_2)\mathbf{n}_2 \times \mathbf{n}_3|_z \\ - Q_j(\xi_1)\mathbf{n}_1\mathbf{n}_3 - Q_j(\xi_2)\mathbf{n}_2\mathbf{n}_3 - Q_j(\xi_3)]\bigg|_{\xi_3 = 0} = T_j, \\ j = 1, 2$$

or, after some simplification using the equalities

$$\frac{\partial}{\partial n_3}F(\xi_j) = F'(\xi_j)\mathbf{n}_j\mathbf{n}_3 = -\frac{1}{2}F'(\xi_j), \\ j = 1, 2, \quad \frac{\partial}{\partial n_3}F(\xi_3) = F'(\xi_3)$$

and properties (3.2) and (3.3)

$$\frac{\sqrt{3}}{4}[P'_j(\xi_1) - P'_j(2h-\xi_1)] - \frac{1}{4}[Q'_j(\xi_1) \\ + Q'_j(2h-\xi_1)] - Q'_j(0) = T_j, \quad j = 1, 2 \tag{3.20}$$

The left-hand sides of equalities (3.20) contain the variable ξ_1, whereas the right-hand sides are constant. In order to remove this obvious contradiction, we must take into account the fact that, after differentiating Eq. (3.11) with respect to ξ_1 we can conclude that in Eq. (3.20) the expressions in both square brackets are equal to zero, i.e. conditions (3.20) do not in fact contain the variable ξ_1 and take the form

$$-Q'_j(0) = T_j, \quad j = 1, 2 \tag{3.21}$$

Hence, boundary conditions (1.3) reduce to four equalities – (3.13) and (3.20). Using relations (2.6) we can write them in the form of a linear closed algebraic system in C_1, C_2, D_1 and D_2. Solving this system we obtain

$$C_j = -[u_{j0}\cos n_0 h\,\mathrm{sh}\,m_0 h + (-1)^j u_{(3-j)0}\sin n_0 h\,\mathrm{ch}\,m_0 h]/\Delta_{1\lambda}$$

$$D_j = [(-1)^j T_{3-j}(p_0\sin q_0 h\,\mathrm{ch}\,p_0 h + q_0\cos q_0 h\,\mathrm{sh}\,p_0 h) + $$
$$+ T_j(p_0\cos q_0 h\,\mathrm{sh}\,p_0 h - q_0\sin q_0 h\,\mathrm{ch}\,p_0 h)]/\Delta_{1\mu}, \quad j = 1, 2$$

$$\Delta_{1\lambda} = \mathrm{ch}\,2m_0 h - \cos 2n_0 h > 0,$$
$$\Delta_{1\mu} = (p_0^2 + q_0^2)[\mathrm{ch}\,2p_0 h - \cos 2q_0 h] > 0 \tag{3.22}$$

All the expressions of the first exact solution of problem (1.2), (1.3), obtained for a viscoelastic rod of triangular cross-section in the first version of the boundary conditions, are extremely lengthy, and hence we will not give its final form, and will merely indicate the number in the sequence of working formulae, which enable this solution to be obtained: the displacements u and v are found from formulae (1.5), the functions U_1, U_2 and V_1, V_2 are found from formulae (3.6), P_1, P_2 and Q_1, Q_2 are found from formulae (2.6), the coefficients C_3, D_3 and C_4, D_4 are found from formulae (3.12), and the coefficients C_1, C_2, D_1, D_2 and $\Delta_{1\lambda}$, Δ_{1m} are found from formulae (3.22). To obtain this solution numerically, all these actions must be carried out in the reverse order. The displacements u and v are expressed in terms of differentiable functions, and hence from the known formulae one can obtain the strains and strain rates, while the stresses can be obtained from relation (1.1).

In the exact solution obtained the oscillatory process is determined by the following six dimensionless parameters

$$\delta_\lambda = (\omega_\lambda\zeta_0/\lambda_0)^2, \quad \delta_\mu = (\omega_\mu\eta/\mu)^2$$

$$R_1 = 2m_0 h = \frac{\omega h}{\Lambda_0}\sqrt{2\rho(\Lambda_0 - \lambda_0)},$$

$$R_2 = 2n_0 h = \frac{\omega h}{\Lambda_0}\sqrt{2\rho(\Lambda_0 + \lambda_0)}$$

$$S_1 = 2p_0 h = \frac{\omega}{G_0}\sqrt{2\rho(G_0 - \mu)},$$

$$S_2 = 2q_0 h = \frac{\omega}{G_0}\sqrt{2\rho(G_0 + \mu)} \tag{3.23}$$

The parameters δ_λ and δ_μ represent the contribution of the viscous properties with respect to the contribution of the elastic properties to the longitudinal and shear harmonic waves, R_1 and S_1 define the nature of the attenuation of the longitudinal and shear oscillations, respectively, while the velocities of motion of the wave fronts v_λ and v_μ and the wavelengths of these waves L_λ and L_μ can be calculated in terms of R_2 and S_2:

$$\upsilon_\lambda = 2\frac{\omega h}{R_2} = \frac{\sqrt{2}\Lambda_0}{\sqrt{\rho(\Lambda_0 + \lambda_0)}} > \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

$$L_\lambda = \frac{2\pi}{n_0} = \frac{2\pi}{\omega}\upsilon_\lambda > \frac{2\pi}{\omega}\sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$\upsilon_\mu = \frac{\omega}{q_0} = 2\frac{\omega h}{S_2} = \frac{\sqrt{2}G_0}{\sqrt{\rho(G_0 + \mu)}} > \sqrt{\frac{\mu}{\rho}},$$

$$L_\mu = \frac{2\pi}{p_0} = \frac{2\pi}{\omega}\upsilon_\mu > \frac{2\pi}{\omega}\sqrt{\frac{\mu}{\rho}} \tag{3.24}$$

It can be seen that the viscosity leads to an increase in the velocities of motion of the wave fronts and the wavelength of the longitudinal and shear harmonic waves compared with their values for an ideal elastic medium.

We can conclude from expressions (2.6) for the functions $P_j(x)$ and $Q_j(x)$ that the parameters $R_1 = 2m_0h$ and $S_1 = 2p_0h$ are the decrement components, while $R_2 = 2n_0h$ and $S_2 = 2q_0h$ are the phase components. In the case of an elastic medium, the resonance frequencies $\omega_{0\lambda}$ and $\omega_{0\mu}$ of the longitudinal and shear waves are found from the equations $\Delta_{1\lambda} = 0$ and $\Delta_{1\mu} = 0$ when $\zeta = n = 0$. For a viscoelastic material these determinants, according to the last two formulae of (3.22), are strictly positive, and hence we can only speak of their minimum. The coefficients C_1, C_2 and D_1, D_2 increase as $\Delta_{1\lambda}$ and $\Delta_{1\mu}$ decrease, and consequently, the amplitudes of the harmonic oscillations also increase, which is observed in experiments. For a viscoelastic medium, the pseudo-resonance frequencies ω_λ and ω_μ define those frequencies for which the determinants $\Delta_{1\lambda}$ and $\Delta_{1\mu}$ take the minimum values in the dependence on the phase components R_2 and S_2, which corresponds to the conditions

$$\cos 2n_0 h = 1, \quad \cos 2q_0 h = 1 \tag{3.25}$$

Hence, using the expressions for n_0 and q_0, defined by the fourth equalities of (2.4) and (2.5), we obtain the following cubic equations in the squares of the pseudo-resonance frequencies ω_λ^0 and ω_μ^0

$$\rho^2 h^4(\lambda_0^2 + \omega_\lambda^2\zeta_0^2)\omega_\lambda^4 = [8k^2\pi^2\lambda_0^2 + (8k^2\pi^2\zeta_0^2 - \rho h^2\lambda_0)\omega_\lambda^2]^2$$
$$\rho^2 h^4(\mu^2 + \omega_\mu^2\eta^2)\omega_\mu^4 = [8k^2\pi^2\mu^2 + (8k^2\pi^2\eta^2 - \rho h^2\mu)\omega_\mu^2]^2 \tag{3.26}$$

If we take the small dimensionless quantities δ_λ and δ_μ for small coefficients of viscosity, then, using binomial expansions of the quantities Λ_0, $\sqrt{\Lambda_0 + \lambda_0}$, G_0, $\sqrt{G_0 + \mu}$, we can obtain estimates for the pseudo-resonance frequencies

$$\omega_\lambda \simeq \omega_{0\lambda}\left(1 + \frac{3}{8}\omega_{0\lambda}^2\frac{\zeta_0^2}{\lambda_0^2}\right) > \omega_{0\lambda} = 2k\frac{\pi}{h}\sqrt{\frac{\lambda_0}{\rho}}$$

$$\omega_\mu \simeq \omega_{0\mu}\left(1 + \frac{3}{8}\omega_{0\mu}^2\frac{\eta^2}{\mu^2}\right) > \omega_{0\mu} = 2k\frac{\pi}{h}\sqrt{\frac{\mu}{\rho}} \tag{3.27}$$

These estimates show that the viscosity leads to an increase in the pseudo-resonance frequencies compared with the resonance frequencies $\omega_{0\lambda}$ and $\omega_{0\mu}$.

4. The solution for a viscoelastic rod of triangular cross-section for the second version of the boundary conditions

For boundary conditions (1.4), we will represent the tangential component of the displacement vector in the form

$$u_\tau|_\Gamma = (u\tau_x + \upsilon\tau_y)|_\Gamma = (un_y - \upsilon n_x)|_\Gamma, \quad \text{or}$$
$$(U_j n_y - V_j n_x)|_\Gamma = \upsilon_{j0}, \quad j = 1, 2 \tag{4.1}$$

The further considerations and calculations are largely similar to those used in Section 3 for the first version of the boundary conditions. Hence, we will only present the main results.

The necessary conditions for an exact solution to exist in the form of a relation between the coefficients C_j and D_j, similar to conditions (3.12), will now have the form

$$C_3 = C_1, \quad C_4 = -C_2, \quad D_3 = -D_1, \quad D_4 = D_2 \tag{4.2}$$

while from the boundary conditions (1.4), like Eq. (3.13), we obtain

$$Q_j(0) = -\upsilon_{j0}, \quad j = 1, 2 \tag{4.3}$$

Hence, the first boundary condition (1.4), written in the form (4.1), will be satisfied if the coefficients C_j and D_j are connected by relations (4.2) and (4.3).

We will convert the second boundary condition of (1.4) for the normal stress. Taking into account the fact that on Γ the component u_τ is assumed to be constant, the normal stress $\sigma_{n|\Gamma}$ can be represented by the expression

$$\sigma_n|_\Gamma = \lambda_0 \frac{\partial u_n}{\partial n}\Big|_\Gamma + \zeta_0 \frac{\partial}{\partial n}\frac{\partial u_n}{\partial t}\Big|_\Gamma = \sigma_{10}\cos\omega t + \sigma_{20}\sin\omega t \tag{4.4}$$

Using representation (1.5) and introducing the notation

$$N_j = \frac{\partial}{\partial n_3}(U_j n_{3x} + V_j n_{3y})\Big|_{\xi_3 = 0}, \quad j = 1, 2 \tag{4.5}$$

we can obtain from Eq.(4.4) a system of two equations for N_j, the solution of which has the form

$$N_j = \frac{\lambda_0\sigma_{j0} + (-1)^j\zeta_0\omega\sigma_{(3-j)0}}{\lambda_0^2 + \zeta_0^2\omega^2}, \quad j = 1, 2 \tag{4.6}$$

If we substitute expressions (3.6) into the right-hand side of boundary conditions (4.5), then, after some simplification, we will have the following conditions, similar to conditions (3.21):

$$P'_j(0) = N_j, \quad j = 1, 2 \tag{4.7}$$

Hence, boundary conditions (1.4) have been reduced to a system of four equations – (4.3) and (4.7). After substituting expressions (2.6) into them we obtain the coefficients C_j and D_j:

$$C_j = [N_j(n_0\sin n_0h\,\mathrm{ch}\,m_0h - m_0\cos n_0h\,\mathrm{sh}\,m_0h) -$$
$$-(-1)^j N_{3-j}(m_0\sin n_0h\,\mathrm{ch}\,m_0h + n_0\cos n_0h\,\mathrm{sh}\,m_0h)]/\Delta_{2\lambda}$$
$$D_j = [\upsilon_{j0}\cos q_0h\,\mathrm{sh}\,p_0h + (-1)^j\upsilon_{(3-j)0}\sin q_0h\,\mathrm{ch}\,p_0h]/\Delta_{2\mu};$$
$$j = 1, 2$$
$$\Delta_{2\lambda} = (m_0^2 + n_0^2)[\mathrm{ch}\,2m_0h - \cos 2n_0h] > 0,$$
$$\Delta_{2\mu} = \mathrm{ch}\,2p_0h - \cos 2q_0h > 0 \tag{4.8}$$

We will indicate the number in the sequence of working formulae which enable one to obtain a solution of the problem for the second version of the boundary conditions: the displacements u and v are found from formulae (1.5), the functions U_1, U_2 and V_1, V_2 are found from formulae (3.6), P_1, P_2 and Q_1, Q_2 are found from formulae (2.6), the coefficients C_3, D_3 and C_4, D_4 are found from formulae (4.2), and the coefficients C_1, C_2, D_1, D_2 and $\Delta_{2\lambda}, \Delta_{2m}$ are found from formulae (4.8). To obtain the solution numerically, all these actions must be carried out in the reverse order.

The oscillatory process is defined by the same six dimensionless parameters (3.23), the pseudo-random frequencies are calculated from Eq. (3.26), and formulae (3.24) for the velocities v_λ and v_μ and the wavelengths L_λ and L_μ of the longitudinal and shear waves remain true in this case also.

It follows from formulae (3.24) that the velocities v_λ and v_μ and the wavelengths L_λ and L_μ at high frequencies may considerably exceed the velocities and wavelengths of the corresponding elastic waves. The longitudinal and shear waves do not influence one another in the exact solutions obtained, the normal actions on the boundary Γ (u_n from conditions (1.3) or σ_n from conditions (1.4)) only produce longitudinal waves, while the shear actions on Γ (τ_n from conditions (1.3) or u_τ from conditions (1.4)) only produce shear waves. No wave nodes, i.e. fixed points in the area of the triangular cross-section of the rod, at which the amplitudes U_j and V_j simultaneously vanish, have been found. Using the analytical formulae for the amplitudes of the displacements (3.6) we can calculate the various quantitative characteristics of the stress and strain fields in the triangular section of a viscoelastic rod, including singular points – its vertices.

The exact solutions obtained may be useful for debugging the programs of approximate methods for solving multidimensional dynamic curvilinear boundary-value problems of viscoelasticity. They can also be used to find the coefficients of viscosity ζ and η, for example, by comparing the velocities v_λ and v_μ for the wavelengths L_λ and L_μ, obtained experimentally and calculated from formulae (3.24). These coefficients are still unknown for many viscoelastic materials.[7]

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